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AXISYMMETRIC THERMAL STRESS IN A THIN CIRCULAR DISK BY THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS

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Abstract—Using the method of matched asymptotic expansions, a solution is constructed for thin disks with stress-free edges which consists of an analytical solution in the interior and a boundary layer correction. The analytical solution is shown to be accurate to order $(h/a)^2$ compared to unity, and is that given by the classical theory of thin plates. There is no boundary layer in the case of a clamped disk, so that the thin plate solution is valid throughout. The solution of the boundary layer equations was obtained numerically by solving the finite difference form of the governing equations using an iterative scheme called "Dynamic Relaxation".

INTRODUCTION

As is well known, it is not possible to obtain a general solution of the equations of linear thermoelasticity for a circular disk containing an axisymmetric temperature distribution. The problem seems to lie in the inability of all methods to satisfy general boundary conditions simultaneously on the lateral ($z = 0, h$) and edge faces ($r = a$). Though numerical methods such as the Finite Element Method have been available for some time, they are essentially "ad hoc" in character and are generally not useful in predicting solutions for other temperature distributions than the one computed. Clearly, if some analytical predictions could be made, it would be helpful to the analyst and would also reduce the amount of computation necessary to evaluate a design.

Some progress toward obtaining analytical solutions can be made if the disk can be considered thin ($h/a \ll 1$). For this case, Friedrichs and Dressler [1] expressed the solution in an asymptotic series using the thickness (h) as the expansion parameter, and obtained the governing equations at each stage by taking appropriate limits of the general equations. They showed that the solution for a laterally loaded rectangular plate is essentially that predicted by the classical theory of plates in the interior region, and has a boundary layer behavior near the edge $r = a$. The boundary layer solution is governed by equations analogous to the problem of two-dimensional plane strain and must be obtained numerically. It should be noted that the existence and importance of such a layer had been noted earlier by Reissner and Johnson [2], but no solutions of the boundary layer equations were proposed as they were primarily interested in the form of the interior solution.

The basic asymptotic representation of the solution that was used by both [1, 2] was subsequently applied by Reiss [3] and Reiss and Locke [4] to study the essentially different effects of bending and stretching of a laterally loaded circular plate. As the edge in this case was a curve, this gave an indication as to how the analysis of [1] could be extended to other than rectangular shapes, and showed that only one coordinate in a general boundary layer analysis need be stretched.

The method of analysis followed here is similar to that of [3]. Though the thermal problem is closely related to the mechanical problem, it differs in that the equivalent mechanical problem requires a body force distribution that was not considered in [3]. Thus, rather than augment Reiss' analysis, we proceed to formulate the thermal problem from first principles. As in [3], we express the solution as an asymptotic expansion in the thinness parameter (h/a), and proceed to obtain a hierarchy of equations describing separately the state of stress in the interior of the disk and in the "Elasticity" layer. As was noted in [1, 3], we find that the zeroth order equations for the interior region are identical to those of classical plate theory, and the boundary layer problems can be formulated as equivalent plane strain problems. Finally, we verify the validity of the expansion process and determine the equivalent boundary conditions of the interior problem using the matching process described by Cole [5].

Numerical results are also presented for a temperature distribution that has a constant curvature in the z-direction at $r = a$. These results are considered noteworthy, as this study was originally motivated by a desire to determine if the thin plate stress distribution generally used (for example, see Ref. [6]) to correlate experimental data in thermal shock studies could be in error near the edge. In such tests, the plate is heated on one or both lateral sides by a hot gas or a high intensity lamp, and is supported along the edge such that there is no radial restraint. It can be shown that the temperature distribution due to lamp radiation is very nearly of constant curvature, and that due to the hot gas can be considered approximately so. As the disks are usually thin, it is generally regarded that the effect of the boundary layer stresses on the maximum tensile and compressive stresses predicted by plate theory is negligible. It is one of the primary purposes of this paper to investigate the effects of this assumption.

DIMENSIONLESS FORM OF THE GOVERNING EQUATIONS

The equations governing the stresses and displacements in a full circular disk due to an axisymmetric temperature distribution are given by Johns [7]. In dimensionless form, using the following dimensionless variables

$$r = a\bar{r} \qquad z = h\bar{z} \qquad T = T_0 \cdot \bar{T}(\bar{r}, \bar{z})$$
$$(u, w) = \alpha a T_0 \cdot (\bar{u}, a\bar{w}/h)$$
$$(\sigma_{rr}, \sigma_{\theta\theta}, \tau_{rz}, \sigma_{zz}) = = \alpha E T_0 (\bar{\sigma}_r, \bar{\sigma}_\theta, h\bar{\tau}/a, h^2\bar{\sigma}_z/a^2)$$

they are: the two equations of equilibrium

$$\frac{\partial \bar{\sigma}_r}{\partial \bar{r}} + \frac{\partial \bar{\tau}}{\partial \bar{z}} + (\bar{\sigma}_r - \bar{\sigma}_\theta)/\bar{r} = 0 \tag{1}$$

$$\frac{\partial \bar{\tau}}{\partial \bar{r}} + \frac{\partial \bar{\sigma}_z}{\partial \bar{z}} + \bar{\tau}/\bar{r} = 0 \tag{2}$$

and the four stress-displacement equations

$$\frac{\partial \bar{u}}{\partial \bar{r}} = \bar{T} + \bar{\sigma}_r - \nu\bar{\sigma}_\theta - \nu\bar{h}^2\bar{\sigma}_z \tag{3}$$

$$\bar{u}/\bar{r} = \bar{T} + \bar{\sigma}_\theta - \nu\bar{\sigma}_r - \nu\bar{h}^2\bar{\sigma}_z \tag{4}$$

$$\frac{\partial \bar{w}}{\partial \bar{z}} = \bar{h}^2[\bar{T} + \bar{h}^2\bar{\sigma}_z - \nu \cdot (\bar{\sigma}_r + \bar{\sigma}_\theta)] \tag{5}$$

$$\frac{\partial \bar{u}}{\partial \bar{z}} + \frac{\partial \bar{w}}{\partial \bar{r}} = 2(1+\nu)\bar{h}^2\bar{\tau} \tag{6}$$

where $\bar{h}(= h/a)$ is the thinness parameter, T_0 is a temperature scale factor and α, E, ν are the coefficient of thermal expansion, Young's modulus and Poission's ratio respectively of the material. In the following sections, solutions of these equations will be sought for thin disks that are heated only, i.e. satisfy the stress-free boundary conditions on $z = 0$, h (see Fig. 1) that

$$\sigma_{zz}, \tau_{rz} = 0 \qquad (0 \leq r \leq a)$$

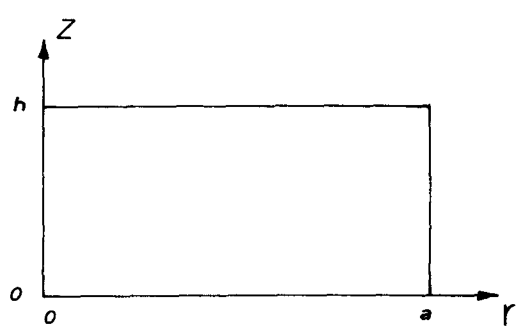

Fig. 1. Coordinate system.

The boundary conditions on the outer edge ($r = a$) will be chosen from either the stress-free condition that

$$\sigma_{rr},\ \tau_{rz} = 0 \qquad (0 \leq z \leq h)$$

or the clamped condition that

$$u, w = 0 \qquad (0 \leq z \leq h).$$

LIMITING SOLUTION IN THE INTERIOR

For very thin disks, the limiting case $\bar{h} \to 0$ seems appropriate, and the above stress-displacement equations can be simplified to yield

$$\frac{\partial \bar{w}_0}{\partial \bar{z}} = 0 \qquad \frac{\partial \bar{u}_0}{\partial \bar{z}} + \frac{\partial \bar{w}_0}{\partial \bar{r}} = 0$$

$$(1-\nu)\cdot \bar{\sigma}_r^{(0)} = -\bar{T} + \left(\frac{\partial \bar{u}_0}{\partial \bar{r}} + \nu \bar{u}/\bar{r}\right)/(1+\nu)$$

$$(1-\nu)\cdot \bar{\sigma}_\theta^{(0)} = -\bar{T} + \left(\nu\frac{\partial \bar{u}_0}{\partial \bar{r}} + \bar{u}_0/\bar{r}\right)/(1+\nu)$$

where the subscript or superscript (0) has been introduced to indicate that this is the approximate solution corresponding to $\bar{h} = 0$. As the transverse normal strain and the shear strain components vanish identically, it immediately follows that

$$\bar{w}_0 = W_0(\bar{r}) \qquad \bar{u}_0 = U_0(\bar{r}) - \bar{z}W_0'(\bar{r}) \qquad (\,)' \equiv \frac{\mathrm{d}}{\mathrm{d}\bar{r}}(\,).$$

Note that this approximate displacement field agrees with that usually assumed *a priori* in the classical theory of thin plates. Further, the in-plane stress components can be rewritten as

$$(1-\nu)\bar{\sigma}_r^{(0)} = -\bar{T} + [U_0' + \nu U_0/\bar{r} - \bar{z}(W_0'' + \nu W_0'/\bar{r})]/(1+\nu)$$
$$(1-\nu)\bar{\sigma}_\theta^{(0)} = -\bar{T} + [\nu U_0' + U_0/\bar{r} - \bar{z}(\nu W_0'' + W_0'/\bar{r})]/(1+\nu)$$

which explicitly show the dependancy of the stresses on the variable $\bar{z}$. It is just this accomplishment which allows the analysis to proceed toward a complete solution.

With the $\bar{z}$-dependancy of the in-plane stresses now known, we can determine the corresponding transverse shear and normal stresses from the equations of equilibrium. Thus, rewriting the equations of equilibrium in terms of U_0, W_0 and making use of the stress-free boundary condition on $\bar{z} = 0$, it follows that

$$(1-\nu)\bar{\tau}_0 = \frac{\partial Z}{\partial \bar{r}} - \bar{z}[(\bar{r}U_0)'/\bar{r} - \bar{z}(\bar{r}W_0')'/2\bar{r}]'/(1+\nu)$$

$$(1-\nu)\bar{\sigma}_z^{(0)} = -\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left[\bar{r}\frac{\partial}{\partial \bar{r}}\int_0^{\bar{z}} Z(\bar{r},x)\,\mathrm{d}x\right] + \frac{\bar{z}^2/2\bar{r}}{1+\nu}\{\bar{r}[(\bar{r}U_0)'/\bar{r} - \bar{z}(\bar{r}W_0')'/3\bar{r}]'\}'$$

where we have introduced the temperature integral $Z(\bar{r}, \bar{z})$ given by

$$Z(\bar{r},\bar{z}) = \int_0^{\bar{z}} \bar{T}(\bar{r},x)\,\mathrm{d}x.$$

The equations governing U_0, W_0 now follow from these stress components by imposing on them the requirement that the surface $\bar{z} = 1$ be stress-free. Thus, in terms of the temperature integral $\bar{T}_m(\bar{r})$, where

$$\bar{T}_m(\bar{r}) = \int_0^1 \bar{T}(\bar{r},x)\,\mathrm{d}x = Z(\bar{r},1)$$

it follows from the expressions for the transverse shear and normal stresses that the displacements U_0, W_0 must satisfy

$$\{(\bar{r}U_0)'/\bar{r} - (\bar{r}W_0')'/2\bar{r} - (1+\nu)\bar{T}_m\}' = 0$$

$$\left\{\bar{r}[(\bar{r}U_0)'/\bar{r} - (\bar{r}W_0')'/3\bar{r}]' - 2(1+\nu)\left[\bar{r}\frac{\mathrm{d}}{\mathrm{d}\bar{r}}\int_0^1 Z(\bar{r},x)\,\mathrm{d}x\right]'\right\}' = 0$$

These equations can readily be integrated to yield

$$\begin{aligned} U_0 &= -A_1\bar{r} + 3\bar{r}A_3/2 + 3\bar{r}A_2(\ln\bar{r} - 1/2)/2 \\ &\quad - 2\frac{1+\nu}{\bar{r}}\int_0^{\bar{r}} \bar{T}_m(x)x\,\mathrm{d}x + 6\frac{1+\nu}{\bar{r}}\int_0^{\bar{r}} x\,\mathrm{d}x\int_0^1 Z(x,y)\,\mathrm{d}y \\ W_0'/6 &= (-A_1 + A_3)\bar{r}/2 + A_2\bar{r}(\ln\bar{r} - 1/2)/2 \\ &\quad - \frac{1+\nu}{\bar{r}}\int_0^{\bar{r}} \bar{T}_m(x)x\,\mathrm{d}x + 2\frac{1+\nu}{\bar{r}}\int_0^{\bar{r}} x\,\mathrm{d}x\int_0^1 Z(x,y)\,\mathrm{d}y. \end{aligned}$$

With the form of the deflection components now determined, we return to the expressions for the stress components and find that

$$\begin{aligned} (1-\nu)\bar{\sigma}_r^{(0)} &= -\bar{T} - 2(1-3\bar{z})\left[A_1/2 + \bar{T}_m(\bar{r}) - \frac{1-\nu}{\bar{r}^2}\int_0^{\bar{r}} \bar{T}_m(x)x\,\mathrm{d}x\right] \\ &\quad + 6(1-2\bar{z})\left[A_3/4 + \int_0^1 Z(\bar{r},x)\,\mathrm{d}x - \frac{1-\nu}{\bar{r}^2}\int_0^{\bar{r}} x\,\mathrm{d}x\int_0^1 Z(x,y)\,\mathrm{d}y\right] \\ (1-\nu)\bar{\sigma}_\theta^{(0)} &= -\bar{T} - 2(1-3\bar{z})\left[A_1/2 + \nu\bar{T}_m(\bar{r}) + \frac{1-\nu}{\bar{r}^2}\int_0^{\bar{r}} \bar{T}_m(x)x\,\mathrm{d}x\right] \\ &\quad + 6(1-2\bar{z})\left[A_3/4 + \nu\int_0^1 Z(\bar{r},x)\,\mathrm{d}x + \frac{1-\nu}{\bar{r}^2}\int_0^{\bar{r}} x\,\mathrm{d}x\int_0^1 Z(x,y)\,\mathrm{d}y\right] \\ (1-\nu)\bar{\tau}_0 &= \left[Z(\bar{r},\bar{z}) - \bar{z}(3\bar{z}-2)\bar{T}_m(\bar{r}) - 6\bar{z}(1-\bar{z})\int_0^1 Z(\bar{r},x)\,\mathrm{d}x\right]' \\ (1-\nu)\bar{\sigma}_z^{(0)} &= -\left\{\bar{r}\left[\int_0^{\bar{z}} Z(\bar{r},x)\,\mathrm{d}x - \bar{z}^2(3-2\bar{z})\int_0^1 Z(\bar{r},x)\,\mathrm{d}x + \bar{z}^2(1-\bar{z})\bar{T}_m(\bar{r})\right]'\right\}'/\bar{r}. \end{aligned}$$

In arriving at these expressions, we have taken $A_2 = 0$ in order to avoid a singularity at $\bar{r} = 0$. The remaining constant A_1, A_3 should, in principle, be determined by the boundary conditions on the edge $\bar{r} = 1$. Note, however, that the transverse shear and normal stresses are completely determined, and vanish identically for temperature distributions that are independent of the radial coordinate. The fact that the shear stress $\bar{\tau}_0$ is completely determined has the important consequence that this solution is unable to satisfy the stress-free boundary condition on $\bar{r} = 1$. This should also be apparent from the form of the in-plane stress components, as no choice of A_1, A_3 could satisfy the stress-free boundary condition on $\bar{r} = 1$ for an arbitrary temperature distribution. It is apparent, then, that this solution is not valid in the neighborhood of the edge, and a boundary layer solution must be imposed if the stress-free boundary condition is to be satisfied.

It is possible, however, to satisy the clamped boundary conditions with the current solution, though we must require that both $U_0(1)$, $W_0'(1)$ vanish in order that $\bar{u}_0(\bar{r} = 1, \bar{z}) = 0$ for all $\bar{z}$. The transverse displacement itself can be made to vanish by appropriately choosing the additional constant which results from integrating $W_0'(\bar{r})$.

Thus, we observe that the clamped boundary condition leads to a relatively simple solution, whereas the stress-free boundary condition can only be satisfied by introducing a boundary layer near the edge. It is possible, however, that a boundary layer might have to be introduced, even for the clamped boundary condition, in order that higher order solutions in the interior region satisfy the required boundary condition. Though this matter is of some theoretical importance, it will not be pursued here and we will concentrate on the more immediate problem of the stress-free boundary condition.

As a final comment, it should be noted that the solution developed above for the displacements and the in-plane stresses is precisely that which is obtained from classical thin plate theory (see Johns [7] for example). Thus, the results of classical plate theory are a very good approximation for the stresses and deflections throughout a thin, clamped plate, but are in some error in the neighborhood of a free edge. The extent to which this error is important is open to question. The possibility of a stress concentration is usually discounted, though it would be helpful if some general characteristics of the solution near a free edge could be determined. In the sections which follow, the form of the solution in the boundary layer will be determined, and an attempt will be made to evaluate the relative importance of the stress level in this region.

THE BOUNDARY LAYER EQUATIONS

As has been shown by Reiss [3] and others, the solution in the neighborhood of the edge $\bar{r} = 1$ is formulated in terms of the "Elasticity" boundary layer coordinate ξ, where

$$r = a - h\xi.$$

Thus, we expect that the solution near the edge will vary significantly over lengths comparable with the thickness h. Now, guided by the solution in the interior region, we assume that the order of magnitude of the displacements in the edge region to be identical to that in the interior region, i.e.

$$(u, w) = \alpha a T_0(\hat{u}, a\hat{w}/h).$$

However, in order that the solution near the edge be capable of varying significantly over lengths comparable with the thickness h, the stress components must all be of the same order of magnitude and hence

$$(\sigma_{rr}, \sigma_{\theta\theta}, \tau_{rz}, \sigma_{zz}) = \alpha E \tau_0(\hat{\sigma}_r, \hat{\sigma}_\theta, \hat{\tau}, \hat{\sigma}_z).$$

Furthermore, for thin disks, we assume that the behavior of the solution on the thinness parameter $\bar{h}$ is of the following form

$$\begin{aligned}
\hat{\sigma}_r(\xi, \bar{z}, \bar{h}) &= \hat{\sigma}_r^{(0)}(\xi, \bar{z}) + \bar{h}\hat{\sigma}_r^{(1)}(\xi, \bar{z}) + \bar{h}^2\hat{\sigma}_r^{(2)}(\xi, \bar{z}) + O(\bar{h}^3)\\
\hat{\sigma}_\theta(\xi, \bar{z}, \bar{h}) &= \hat{\sigma}_\theta^{(0)}(\xi, \bar{z}) + \bar{h}\hat{\sigma}_\theta^{(1)}(\xi, \bar{z}) + \bar{h}^2\hat{\sigma}_\theta^{(2)}(\xi, \bar{z}) + O(\bar{h}^3)\\
\hat{\sigma}_z(\xi, \bar{z}, \bar{h}) &= \hat{\sigma}_z^{(0)}(\xi, \bar{z}) + \bar{h}\hat{\sigma}_z^{(1)}(\xi, \bar{z}) + \bar{h}^2\hat{\sigma}_z^{(2)}(\xi, \bar{z}) + O(\bar{h}^3)\\
\hat{\tau}(\xi, \bar{z}, \bar{h}) &= \hat{\tau}_0(\xi, \bar{z}) + \bar{h}\hat{\tau}_1(\xi, \bar{z}) + \bar{h}^2\hat{\tau}_2(\xi, \bar{z}) + O(\bar{h}^3)\\
\hat{u}(\xi, \bar{z}, \bar{h}) &= \hat{u}_0(\xi, \bar{z}) + \bar{h}\hat{u}_1(\xi, \bar{z}) + \bar{h}^2\hat{u}_2(\xi, \bar{z}) + O(\bar{h}^3)\\
\hat{w}(\xi, \bar{z}, \bar{h}) &= \hat{w}_0(\xi, \bar{z}) + \bar{h}\hat{w}_1(\xi, \bar{z}) + \bar{h}^2\hat{w}_2(\xi, \bar{z}) + \bar{h}^3\hat{w}_3(\xi, \bar{z}) + O(\bar{h}^4)
\end{aligned}$$

The equations governing this solution are obtained by substituting the above forms into eqns (1)–(6) and taking successive limits $\bar{h} \to 0$. When the temperature distribution is also expressed in boundary layer coordinates, it follows that the boundary layer solution must satisfy the following hierarchy of equations:

$$\frac{\partial \hat{\sigma}_r^{(0)}}{\partial \xi} - \frac{\partial \hat{\tau}_0}{\partial \bar{z}} = 0 \qquad \frac{\partial \hat{\sigma}_r^{(1)}}{\partial \xi} - \frac{\partial \hat{\tau}_1}{\partial \bar{z}} = \hat{\sigma}_r^{(0)} - \hat{\sigma}_\theta^{(0)} \tag{7,8}$$

$$\frac{\partial \hat{\tau}_0}{\partial \xi} - \frac{\partial \hat{\sigma}_z^{(0)}}{\partial \bar{z}} = 0 \qquad \frac{\partial \hat{\tau}_1}{\partial \xi} - \frac{\partial \hat{\sigma}_z^{(1)}}{\partial \bar{z}} = \hat{\tau}_0 \tag{9,10}$$

$$\frac{\partial \hat{u}_0}{\partial \xi} = 0 \qquad \frac{\partial \hat{u}_1}{\partial \xi} = -\bar{T}(1, \bar{z}) - \hat{\sigma}_r^{(0)} + \nu(\hat{\sigma}_\theta^{(0)} + \hat{\sigma}_z^{(0)}) \tag{11,12}$$

$$\frac{\partial \hat{u}_2}{\partial \xi} = \xi \frac{\partial \bar{T}}{\partial \bar{r}}(1, \bar{z}) - \hat{\sigma}_r^{(1)} + \nu(\hat{\sigma}_\theta^{(1)} + \hat{\sigma}_z^{(1)}) \tag{13}$$

$$\hat{u}_0 = \bar{T}(1, \bar{z}) + \hat{\sigma}_\theta^{(0)} - \nu(\hat{\sigma}_r^{(0)} + \hat{\sigma}_z^{(0)}) \tag{14}$$

$$\hat{u}_1 = -\xi \frac{\partial \bar{T}}{\partial \bar{r}}(1, \bar{z}) + \hat{\sigma}_\theta^{(1)} - \nu(\hat{\sigma}_r^{(1)} + \hat{\sigma}_z^{(1)}) + \xi \cdot [-\bar{T}(1, \bar{z}) - \hat{\sigma}_\theta^{(0)} + \nu(\hat{\sigma}_r^{(0)} + \hat{\sigma}_z^{(0)}) \tag{15}$$

$$\frac{\partial \hat{w}_0}{\partial \bar{z}} = 0 \qquad \frac{\partial \hat{w}_1}{\partial \bar{z}} = 0 \tag{16, 17}$$

$$\frac{\partial \hat{w}_2}{\partial \bar{z}} = \bar{T}(1, \bar{z}) + \hat{\sigma}_z^{(0)} - \nu(\hat{\sigma}_r^{(0)} + \hat{\sigma}_\theta^{(0)}) \tag{18}$$

$$\frac{\partial \hat{w}_3}{\partial \bar{z}} = -\xi \frac{\partial \bar{T}}{\partial \bar{r}}(1, \bar{z}) + \hat{\sigma}_z^{(1)} - \nu(\hat{\sigma}_r^{(1)} + \hat{\sigma}_\theta^{(1)}) \tag{19}$$

$$\frac{\partial \hat{w}_0}{\partial \xi} = 0 \qquad \frac{\partial \hat{u}_0}{\partial \bar{z}} - \frac{\partial \hat{w}_1}{\partial \xi} = 0 \tag{20, 21}$$

$$\frac{\partial \hat{u}_1}{\partial \bar{z}} - \frac{\partial \hat{w}_2}{\partial \xi} = 2(1+\nu)\hat{\tau}_0 \tag{22}$$

$$\frac{\partial \hat{u}_2}{\partial \bar{z}} - \frac{\partial \hat{w}_3}{\partial \xi} = 2(1+\nu)\hat{\tau}_1 \tag{23}$$

The boundary conditions governing the solution of this system of equations are obtained by substituting the assumed form of the solution into the stress-free boundary conditions and taking successive limits $\bar{h} \to 0$. The result of this procedure is that

$$\hat{\sigma}_r^{(0)}, \hat{\sigma}_r^{(1)}, \ldots, \hat{\tau}_0, \hat{\tau}_1, \ldots = 0 \quad \text{on} \quad \xi = 0$$

and,

$$\hat{\sigma}_z^{(0)}, \hat{\sigma}_z^{(1)}, \ldots, \hat{\tau}_0, \hat{\tau}_1, \ldots = 0 \quad \text{on } \bar{z} = 0,1.$$

In addition to the boundary conditions, the boundary layer solution must match the solution in the interior in some sense as $\bar{h} \to 0$. This matching procedure will not only serve to determine the constants of integration A_1, A_3 of the first order interior solution, but will also fix the order of the correction to the first order interior solution. The details of the matching procedure are deferred to a later section and applied to each of the boundary layer solutions as they are determined.

FIRST ORDER BOUNDARY LAYER SOLUTION

The solution of the boundary layer equations can most conveniently be formulated in terms of the stresses $\hat{\sigma}_r^{(0)}$, $\hat{\sigma}_z^{(0)}$, $\hat{\tau}_0$ to form an equivalent plane strain problem. As a preliminary, however, the displacements $\hat{u}_0$, $\hat{w}_0$, $\hat{w}_1$ must first be determined, and a compatibility equation derived from the equations relating $\hat{u}_1$, $\hat{w}_2$ to augment the two existing equilibrium equations.

The required displacements follow immediately from eqns (11), (16), (17), (20), (21) which essentially require that the radial strain, the transverse normal strain and the shear strain components vanish identically. Thus, it follows that

$$\hat{w}_0 = \text{const.} \quad \hat{u}_0 = \hat{u}_0(\bar{z}) \quad \hat{w}_1 = \hat{w}_1(\xi) \quad \frac{d\hat{u}_0}{d\bar{z}} = \frac{d\hat{w}_1}{d\xi} = \text{const.}$$

In anticipation of the first order matching requirement involving $\bar{w}_0$, $\hat{w}_0$, and $\bar{u}_0$, $\hat{u}_0$, the constants can also be evaluated to yield

$$\hat{u}_0 = U_0(1) - \bar{z}W_0'(1) \quad \hat{w}_0 = W_0(1) \quad \frac{d\hat{w}_1}{d\xi} = -W_0'(1) \tag{24}$$

Furthermore, the "hoop" stress component $\hat{\sigma}_\theta^{(0)}$ can be evaluated from the "hoop" strain expression (eqn 14) and the displacements to obtain

$$\hat{\sigma}_\theta^{(0)} = U_0(1) - \bar{z}W_0'(1) - \bar{T}(1, \bar{z}) + \nu \cdot (\hat{\sigma}_r^{(0)} + \hat{\sigma}_z^{(0)}) \tag{25}$$

We now proceed to obtain a compatibility equation by successively eliminating the displacement components $\hat{u}_1$, $\hat{w}_2$ from eqns (12), (18), (22). Thus, on taking the appropriate derivatives, there results the following compatibility equation

$$\frac{\partial^2}{\partial \bar{z}^2}[\hat{\sigma}_r^{(0)} - \nu(\hat{\sigma}_\theta^{(0)} + \hat{\sigma}_z^{(0)})] + \frac{\partial^2}{\partial \xi^2}[\hat{\sigma}_z^{(0)} - \nu(\hat{\sigma}_r^{(0)} + \hat{\sigma}_\theta^{(0)})] = -2(1+\nu)\frac{\partial^2 \hat{\tau}_0}{\partial \xi \partial \bar{z}} - \frac{\partial^2 \bar{T}}{\partial \bar{z}^2}(1, \bar{z}). \quad (26)$$

It should be noted that this equation could have alternatively been derived directly from the appropriate equation given by Lur'ye[8] for axisymmetric deformation.

At this stage, the displacements are expressed in terms of the interior solution, the "hoop" stress is expressed in terms of the displacements and the stresses $\hat{\sigma}_r^{(0)}$, $\hat{\sigma}_z^{(0)}$, and the stresses $\hat{\sigma}_r^{(0)}$, $\hat{\sigma}_z^{(0)}$, $\hat{\tau}_0$ themselves are determined by the two equations of equilibrium (eqns 7 and 9) and the compatibility equation given above. This situation is analogous to a classical plane strain problem and therefore may be solved in a similar manner.

As is well known from classical theory of elasticity, the equations of equilibrium are satisfied identically if the stresses $\hat{\sigma}_r^{(0)}$, $\hat{\sigma}_z^{(0)}$, $\hat{\tau}_0$ are expressed in terms of an Airy stress function $\phi^{(0)}$, where

$$\hat{\sigma}_r^{(0)} = \frac{\partial^2 \phi^{(0)}}{\partial \bar{z}^2} \qquad \hat{\sigma}_z^{(0)} = \frac{\partial^2 \phi^{(0)}}{\partial \xi^2} \qquad \hat{\tau}_0 = \frac{\partial^2 \phi^{(0)}}{\partial \xi \partial \bar{z}}.$$

When these forms and $\hat{\sigma}_\theta^{(0)}$, as defined by eqn (25), are substituted into the equation of compatibility, there results the following equation for the determination of $\phi^{(0)}$:

$$\nabla^4 \phi^{(0)} = \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \bar{z}^2}\right)^2 \phi^{(0)} = -\frac{1}{1-\nu} \cdot \frac{\partial^2 \bar{T}}{\partial \bar{z}^2}(1, \bar{z}). \quad (27)$$

The stress-free boundary conditions can also be expressed in terms of $\phi^{(0)}$, and lead to the following equivalent representation

$$\phi^{(0)} = 0 \qquad \frac{\partial \phi^{(0)}}{\partial \xi} = 0 \qquad \text{on } \xi = 0, 0 \leq \bar{z} \leq 1$$

$$\phi^{(0)} = 0 \qquad \frac{\partial \phi^{(0)}}{\partial \bar{z}} = 0 \qquad \text{on } \bar{z} = 0, 1, 0 \leq \xi < \infty.$$

Thus, the problem of determining the stress distribution in the boundary layer is analogous to the problem of determining the transverse displacement in a laterally loaded, clamped rectangular strip. It is just this analogy that will be exploited in obtaining numerical solutions. The details of the process, however, are deferred to a later section as it is expedient to first formulate the matching conditions and also the analogous equations governing the second order boundary layer solution.

As a final comment in this section, it should be noted that the existence of the stress function $\phi^{(0)}$ implies that the radial force and moment stress resultants familiar in plate theory vanish everywhere in the boundary layer, i.e.

$$\int_0^1 \hat{\sigma}_r^{(0)} \, d\bar{z} = 0 \qquad \int_0^1 \hat{\sigma}_r^{(0)} \bar{z} \, d\bar{z} = 0. \quad (28)$$

It is these conditions, used in conjunction with the matching principle, that allows us to determine the constants of integration A_1, A_3.

MATCHING CONDITIONS FOR THE FIRST ORDER BOUNDARY LAYER SOLUTION

In general, the matching process described here is that due to Cole [5]. The essential feature of this process is the requirement that the interior and boundary layer expansions for a given dependent variable agree asymptotically in an intermediate region when both expansions are expressed in terms of an appropriate intermediate range variable. Thus, in terms of the intermediate radial coordinate $\tilde{r}$, where

$$\bar{r} = 1 - \eta(\bar{h})\tilde{r} \qquad \xi = \eta(\bar{h})\tilde{r}/\bar{h}$$

and $\eta(\bar{h})$ is such that

$$L_{\bar{h}\to 0}(\eta, \eta/\bar{h}, \bar{h}/\eta) = (0, \infty, 0)$$

we require

$$L_{\bar{h}\to 0,\tilde{r}-\text{fixed}}\{\hat{\sigma}_r(\xi, \bar{z}, \bar{h}) - \bar{\sigma}_r(\bar{r}, \bar{z}, \bar{h})\} = 0 \tag{29}$$

$$L_{\bar{h}\to 0,\tilde{r}-\text{fixed}}\{\hat{\sigma}_\theta(\xi, \bar{z}, \bar{h}) - \bar{\sigma}_\theta(\bar{r}, \bar{z}, \bar{h})\} = 0 \tag{30}$$

$$L_{\bar{h}\to 0,\tilde{r}-\text{fixed}}\{\hat{\sigma}_z(\xi, \bar{z}, \bar{h}) - \bar{h}^2\bar{\sigma}_z(\bar{r}, \bar{z}, \bar{h})\} = 0 \tag{31}$$

$$L_{\bar{h}\to 0,\tilde{r}-\text{fixed}}\{\hat{\tau}(\xi, \bar{z}, \bar{h}) - \bar{h}\bar{\tau}(\bar{r}, \bar{z}, \bar{h})\} = 0 \tag{32}$$

$$L_{\bar{h}\to 0,\tilde{r}-\text{fixed}}\{\hat{u}(\xi, \bar{z}, \bar{h}) - \bar{u}(\bar{r}, \bar{z}, \bar{h})\} = 0 \tag{33}$$

$$L_{\bar{h}\to 0,\tilde{r}-\text{fixed}}\{\hat{w}(\xi, \bar{z}, \bar{h}) - \bar{w}(\bar{r}, \bar{z}, \bar{h})\} = 0 \tag{34}$$

to all orders of magnitude of $\eta(\bar{h})$. Note also that $L_{\bar{h}\to 0,\tilde{r}-\text{fixed}}(\bar{r}, \xi) = (1, \infty)$.

The matching process for the first order terms is particularly simple, as it is necessary only to verify that

$$\hat{\sigma}_r^{(0)}(\xi \to \infty, \bar{z}) = \bar{\sigma}_r^{(0)}(1, \bar{z})$$
$$\hat{\sigma}_\theta^{(0)}(\xi \to \infty, \bar{z}) = \bar{\sigma}_\theta^{(0)}(1, \bar{z})$$
$$\hat{\sigma}_z^{(0)}(\xi \to \infty, \bar{z}) = 0 \qquad \hat{\tau}_0(\xi \to \infty, \bar{z}) = 0$$
$$\hat{u}_0(\xi \to \infty, \bar{z}) = \bar{u}_0(1, \bar{z}) = U_0(1) - \bar{z}W_0'(1)$$
$$\hat{w}_0(\xi \to \infty, \bar{z}) = \bar{w}_0(1, \bar{z}) = W_0(1).$$

The latter two requirements have already been anticipated and used in eqn (24) to define the constants obtained when solving the boundary layer equations for $\hat{u}_0$, $\hat{w}_0$.

Before actually verifying the matching conditions on the stress components, we evaluate the constants A_1, A_3 by using the first requirement above and eqns (28) that noted that the radial force and moment stress resultants vanish everywhere for the stress-free boundary condition. Thus, the observation that

$$\int_0^1 \mathrm{d}\bar{z}\bar{\sigma}_r^{(0)}(1, \bar{z})[1, \bar{z}] = [0, 0]$$

leads to

$$A_1 = 2(1-\nu)\int_0^1 \bar{T}_m(x)x\,\mathrm{d}x \qquad A_3 = 4(1-\nu)\int_0^1 x\,\mathrm{d}x\int_0^1 Z(x, y)\,\mathrm{d}y$$

with the consequences that

$$U_0(1) = -4\int_0^1 \bar{T}_m(x)x\,\mathrm{d}x + 12\int_0^1 x\,\mathrm{d}x\int_0^1 Z(x, y)\,\mathrm{d}y \tag{35}$$

$$W_0'(1) = -12\int_0^1 \bar{T}_m(x)x\,\mathrm{d}x + 24\int_0^1 x\,\mathrm{d}x\int_0^1 Z(x, y)\,\mathrm{d}y \tag{36}$$

$$\hat{\sigma}_\theta^{(0)}(\xi, \bar{z}) = -4(1-3\bar{z})\int_0^1 \bar{T}_m(x)x\,\mathrm{d}x + 12(1-2\bar{z})\int_0^1 x\,\mathrm{d}x\int_0^1 Z(x, y)\,\mathrm{d}y - \bar{T}(1, \bar{z}) + \nu(\hat{\sigma}_r^{(0)} + \hat{\sigma}_z^{(0)}) \tag{37}$$

It also follows that

$$(1-\nu)\bar{\sigma}_\theta^{(0)}(1,\bar{z}) = -\bar{T}(1,\bar{z}) - 4(1-3\bar{z})\left[(1-\nu)\int_0^1 \bar{T}_m(x)x\,\mathrm{d}x + \frac{\nu}{2}\bar{T}_m(1)\right] + 12(1-2\bar{z})\left[(1-\nu)\int_0^1 x\,\mathrm{d}x\int_0^1 Z(x,y)\,\mathrm{d}y + \frac{\nu}{2}\int_0^1 Z(1,y)\,\mathrm{d}y\right] \quad (38)$$

$$(1-\nu)\bar{\sigma}_r^{(0)}(1,\bar{z}) = -\bar{T}(1,\bar{z}) - 2(1-3\bar{z})\bar{T}_m(1) + 6(1-2\bar{z})\int_0^1 Z(1,y)\,\mathrm{d}y. \quad (39)$$

We now take up the task of verifying the limiting behavior of the stress components. The task is rendered almost trivial by a result given previously by Knowles [9], and the observation that a particular solution satisfying the differential equation for $\phi^{(0)}$ (eqn 27) and the boundary conditions on $\bar{z} = 0, 1$ can be taken in the form

$$-(1-\nu)\phi_{\text{part}}^{(0)} = \int_0^{\bar{z}} \mathrm{d}t \int_0^t \bar{T}(1,x)\,\mathrm{d}x + \bar{z}^2(1-\bar{z})\int_0^1 \bar{T}(1,x)\,\mathrm{d}x - \bar{z}^2(3-2\bar{z})\int_0^1 \mathrm{d}t\int_0^t \bar{T}(1,x)\,\mathrm{d}x$$

Note also that

$$\phi_{\text{part}}^{(0)} = \phi_{\text{part}}^{(0)}(\bar{z})$$

and

$$\int_0^1 \mathrm{d}\bar{z}\frac{\mathrm{d}^2\phi_{\text{part}}^{(0)}}{\mathrm{d}\bar{z}^2}[1,\bar{z}] = [0,0].$$

We first consider the solution for $\phi^{(0)}$ to be the superposition of this particular solution and a correction, where the stress distribution due to the correction can be interpreted as that due to a self-equilibrating radial stress distributed over the end $\bar{r} = 1, 0 \leq \bar{z} \leq 1$. Next, following Knowles, we observe that the stress distribution due to the correction must decay exponentially with distance from the end $\bar{r} = 1$, and hence

$$\hat{\sigma}_r^{(0)}(\xi\to\infty,\bar{z}) = \frac{\mathrm{d}^2\phi_{\text{part}}^{(0)}}{\mathrm{d}\bar{z}^2} + \text{T.S.T.}: \quad \hat{\sigma}_z^{(0)}(\xi\to\infty,\bar{z}),\ \hat{\tau}_0(\xi\to\infty,\bar{z}) = \text{T.S.T.}$$

where T.S.T. indicates that the term is transcendentally small. This observation immediately verifies the limiting behavior of $\hat{\sigma}_z^{(0)}$, $\hat{\tau}_0$, and the limiting behavior of $\hat{\sigma}_r^{(0)}$, $\hat{\sigma}_\theta^{(0)}$ follows directly from the relations given above after substituting the expression for $\mathrm{d}^2\phi_{\text{part}}^{(0)}/\mathrm{d}\bar{z}^2$.

Thus, it has been shown that the boundary layer solution given above correctly matches the solution given previously as being valid in the interior of the plate. At this stage, an approximate solution to the problem could be constructed from the solutions given above. However, the question of overall accuracy would remain unanswered, as it has not yet been established what the order of the correction is to the solution in the interior. In order to establish this order, we proceed now to formulate the solution to the second order boundary layer equations.

SECOND ORDER BOUNDARY LAYER SOLUTION

As with the first order equations, the solution to the second order boundary layer equations can be formulated as an equivalent plane strain problem. However, as the two equations of equilibrium are not homogeneous, the equivalent plane strain problem cannot be formulated directly in terms of $\hat{\sigma}_r^{(1)}$, $\hat{\sigma}_z^{(1)}$, $\hat{\tau}_1$ but must alternatively be expressed in terms of a particular solution of the equations of equilibrium and a correction. It follows that the stresses associated with the correction do satisfy equations which are analogous to the plane strain problem. In addition, as with the first order equations, a compatibility equation must also be derived from the equations relating $\hat{u}_2$, $\hat{w}_3$ to augment the two existing equilibrium equations.

A particular solution of the equations of equilibrium (eqns 8, 10) can be formulated from the observation that the equations, when expressed in terms of the stress function $\phi^{(0)}$ and the displacements (U_0, W_0) are similar to those governing a thin beam of rectangular cross-section acted upon by a body force distribution that is independent of the axial coordinate. Thus, when written in the form

$$\frac{\partial}{\partial\xi}\left[\hat{\sigma}_r^{(1)}+\nu\frac{\partial\phi^{(0)}}{\partial\xi}\right]-\frac{\partial}{\partial\bar{z}}\left[\hat{\tau}_1+(1-\nu)\frac{\partial\phi^{(0)}}{\partial\bar{z}}\right]=-U_0(1)+\bar{z}W_0'(1)+\bar{T}(1,\bar{z}) \tag{40}$$

$$\frac{\partial}{\partial\bar{z}}\left[\hat{\sigma}_z^{(1)}+(2-\nu)\frac{\partial\phi^{(0)}}{\partial\xi}\right]-\frac{\partial}{\partial\xi}\left[\hat{\tau}_1+(1-\nu)\frac{\partial\phi^{(0)}}{\partial\bar{z}}\right]=0 \tag{41}$$

the terms in [ ]-brackets appear analogous to the stresses in a thin beam. Now, guided by beam theory, we define the terms in [ ]-brackets to be the stress components $\hat{\sigma}_r^{(b)}$, $\hat{\tau}_b$, $\hat{\sigma}_z^{(b)}$ in a beam, and assume

$$\hat{\sigma}_r^{(b)}=f(\xi)+\bar{z}g(\xi).$$

With this assumption, the form of $\hat{\tau}_b$ satisfying the boundary conditions on $\bar{z}=0,1$ can be obtained by integrating eqn (40), and the form of $\hat{\sigma}_z^{(b)}$ satisfying the boundary conditions on $\bar{z}=0,1$ can be obtained by integrating eqn (41). In particular, it is found that $\hat{\sigma}_z^{(b)}\propto f''(\xi), g''(\xi)$. However, the requirement that $\hat{\tau}_b(\bar{z}=1)=0$ also implies that $f'(\xi)$, $g'(\xi)$ are constants. Thus, it follows that $\hat{\sigma}_z^{(b)}=0$ and f, g are linear in ξ. If we further choose the constants in f, g to be zero, then $\hat{\sigma}_r^{(b)}(0,\bar{z})=0$ and we obtain

$$\hat{\sigma}_r^{(b)}=\xi\cdot\{C(1-2\bar{z})+2\bar{z}[-U_0(1)+W_0'(1)/2+\bar{T}_m(1)]\} \tag{42}$$

$$\hat{\tau}_b=\bar{z}(1-\bar{z})[C+U_0(1)]+\bar{z}^2\bar{T}_m(1)-Z(1,\bar{z}) \tag{43}$$

The constant C is determined by the requirement that the resultant transverse shear force over a typical cross-section vanish, i.e.

$$\int_0^1\hat{\tau}_b\,d\bar{z}=\left[C+U_0(1)+2\bar{T}_m(1)-6\int_0^1 Z(1,x)\,dx\right]/6=0.$$

This completes the formulation of the required particular solution. We now express the second order solution in terms of the components of the correction as

$$\hat{\sigma}_r^{(1)}=\overset{*}{\sigma}_r^{(1)}-\nu\frac{\partial\phi^{(0)}}{\partial\xi}+\hat{\sigma}_r^{(b)}\qquad \hat{\sigma}_z^{(1)}=\overset{*}{\sigma}_z^{(1)}-(2-\nu)\frac{\partial\phi^{(0)}}{\partial\xi}$$

$$\hat{\tau}_1=\overset{*}{\tau}_1-(1-\nu)\frac{\partial\phi^{(0)}}{\partial\bar{z}}+\hat{\tau}_b \tag{44}$$

where (*) is used to denote the stresses associated with the correction. It can readily be verified that the starred stress components satisfy homogeneous equations of equilibrium.

The required compatibility equation is obtained by successively eliminating the displacement components $\hat{u}_2$, $\hat{w}_3$ from the expressions for the second order shear strain (eqn 23), radial normal strain (eqn 13), and transverse normal strain components (eqn 19). Thus, on taking the appropriate derivatives and substituting, we obtain the following compatibility equation

$$\frac{\partial^2}{\partial\bar{z}^2}[\hat{\sigma}_r^{(1)}-\nu(\hat{\sigma}_\theta^{(1)}+\hat{\sigma}_z^{(1)})]+\frac{\partial^2}{\partial\xi^2}[\hat{\sigma}_z^{(1)}-\nu(\hat{\sigma}_r^{(1)}+\hat{\sigma}_\theta^{(1)})]=-2(1+\nu)\frac{\partial^2\hat{\tau}_1}{\partial\xi\partial\bar{z}}+\xi\frac{\partial^3\bar{T}}{\partial\bar{r}\partial\bar{z}^2}(1,\bar{z}). \tag{45}$$

As before, this equation could have been derived directly from the equation given by Lur'ye [8].

However, before the compatibility equation can be used to formulate the equivalent plane strain problem, we must first eliminate the "hoop" stress component $\hat{\sigma}_\theta^{(1)}$, where $\hat{\sigma}_\theta^{(1)}$ is expressed in terms of the radial displacement $\hat{u}_1$ by eqn (15). Thus, with

$$\hat{\sigma}_\theta^{(1)}=\hat{u}_1+\xi\left[\frac{\partial\bar{T}}{\partial\bar{r}}(1,\bar{z})+U_0(1)-\bar{z}W_0'(1)\right]+\nu(\hat{\sigma}_r^{(1)}+\hat{\sigma}_z^{(1)})$$

and using eqns (12), (18), (22) to eliminate the displacements, it follows that

$$\nabla^2\hat{\sigma}_\theta^{(1)} = \nu\nabla^2(\hat{\sigma}_r^{(1)} + \hat{\sigma}_z^{(1)}) + (1+\nu)\left[\frac{\partial}{\partial\xi}(\hat{\sigma}_z^{(0)} - \hat{\sigma}_r^{(0)}) + 2\frac{\partial\hat{\tau}_0}{\partial\bar{z}}\right] + \xi\frac{\partial^3\bar{T}}{\partial\bar{r}\partial\bar{z}^2}(1,\bar{z})$$

and eqn (45) becomes

$$\left[(1-\nu)\frac{\partial^2}{\partial\xi^2} - \nu\frac{\partial^2}{\partial\bar{z}^2}\right]\hat{\sigma}_z^{(1)} + \left[(1-\nu)\frac{\partial^2}{\partial\bar{z}^2} - \nu\frac{\partial^2}{\partial\xi^2}\right]\hat{\sigma}_r^{(1)} + 2\frac{\partial^2\hat{\tau}_1}{\partial\xi\partial\bar{z}} = \xi\frac{\partial^3\bar{T}}{\partial\bar{r}\partial\bar{z}^2}(1,\bar{z}) + \nu\frac{\partial}{\partial\xi}(\nabla^2\phi^{(0)}). \tag{46}$$

The final form of the compatibility equation is obtained by substituting the expressions (eqns 44) for the components of the correction. When this is done, it follows that the second order solution is governed by two homogeneous equations of equilibrium and the compatibility equation given by

$$\left[(1-\nu)\frac{\partial^2}{\partial\xi^2} - \nu\frac{\partial^2}{\partial\bar{z}^2}\right]\overset{*}{\sigma}_z^{(1)} + \left[(1-\nu)\frac{\partial^2}{\partial\bar{z}^2} - \nu\frac{\partial^2}{\partial\xi^2}\right]\overset{*}{\sigma}_r^{(1)} + 2\frac{\partial^2\overset{*}{\tau}_1}{\partial\xi\partial\bar{z}} = \xi\frac{\partial^3\bar{T}}{\partial\bar{r}\partial\bar{z}^2}(1,\bar{z}) + 2(1-\nu)\frac{\partial}{\partial\xi}(\nabla^2\phi^{(0)}). \tag{47}$$

Before proceeding, let us formulate the solution for the radial displacement $\hat{u}_1$ and hence establish the expression for the second order "hoop" stress. As eqn (12) can be written as

$$\frac{\partial}{\partial\xi}\left[\hat{u}_1 - \nu(1+\nu)\frac{\partial\phi^{(0)}}{\partial\xi}\right] = -(1+\nu)\bar{T}(1,\bar{z}) - (1-\nu^2)\hat{\sigma}_r^{(0)} + \nu\hat{u}_0(\bar{z})$$

we can integrate to obtain

$$\hat{u}_1 = \hat{u}_1(0,\bar{z}) + \xi[\nu\hat{u}_0(\bar{z}) - (1+\nu)\bar{T}(1,\bar{z})] - (1-\nu^2)\int_0^\xi \hat{\sigma}_r^{(0)}(x,\bar{z})\,dx + \nu(1+\nu)\frac{\partial\phi^{(0)}}{\partial\xi}$$

where the function of integration $\hat{u}_1(0,\bar{z})$ is obtained later from the matching conditions.

This equation for the displacement $\hat{u}_1$ can be expressed more conveniently for calculation purposes by rewriting the integral of the radial stress in terms of a bounded part as $\xi\to\infty$ and a remainder. This task is accomplished by recalling that the stress function has been defined previously in terms of a particular solution $\phi^{(0)}_{\text{part}}$ and a correction, where the stresses due to the correction are expected to decay exponentially with ξ. Thus, if we define the bounded part of the integral as $F^{(0)}(\xi,\bar{z})$, the remainder is simply the integral of the particular solution and we obtain

$$(1-\nu)\int_0^\xi \hat{\sigma}_r^{(0)}(x,\bar{z})\,dx = F^{(0)}(\xi,\bar{z}) - \xi\left[\bar{T}(1,\bar{z}) + 2(1-3\bar{z})\bar{T}_m(1) - 6(1-2\bar{z})\int_0^1 Z(1,t)\,dt\right].$$

Finally, the expressions for the radial displacement and the "hoop" stress become

$$\begin{aligned}\hat{u}_1 = {}& \hat{u}_1(0,\bar{z}) + (1+\nu)\left[\nu\frac{\partial\phi^{(0)}}{\partial\xi} - F^{(0)}(\xi,\bar{z})\right] + \xi\Bigg\{(1+\nu)\Bigg[2(1-3\bar{z})\bar{T}_m(1)\\ & - 6(1-2\bar{z})\int_0^1 Z(1,t)\,dt\Bigg] + 4\nu\Bigg[-(1-3\bar{z})\int_0^1 \bar{T}_m(x)x\,dx\\ & + 3(1-2\bar{z})\int_0^1 x\,dx\int_0^1 Z(x,y)\,dy\Bigg]\Bigg\}\end{aligned} \tag{48}$$

$$\begin{aligned}\hat{\sigma}_\theta^{(1)} = {}& \hat{u}_1(0,\bar{z}) + \nu(\hat{\sigma}_r^{(1)} + \hat{\sigma}_z^{(1)}) + \nu(1+\nu)\frac{\partial\phi^{(0)}}{\partial\xi} - (1+\nu)F^{(0)}(\xi,\bar{z}) + \xi\frac{\partial\bar{T}}{\partial\bar{r}}(1,\bar{z})\\ & + (1+\nu)\xi\Bigg[2(1-3\bar{z})\bar{T}_m(1) - 6(1-2\bar{z})\int_0^1 Z(1,t)\,dt\\ & - 4(1-3\bar{z})\int_0^1 \bar{T}_m(x)x\,dx + 12(1-2\bar{z})\int_0^1 x\,dx\int_0^1 Z(x,y)\,dy\Bigg].\end{aligned} \tag{49}$$

It should be noted that the accompanying transverse displacement $\hat{w}_1$ has already (eqn 24) been determined along with the first order boundary layer equations, through actually only the slope $(\partial\hat{w}_1/\partial\xi)$ has been computed.

Returning now to the equations governing the stress correction, we proceed, as before, to define the components in terms of an Airy stress function $\phi^{(1)}$, where

$$\overset{*}{\sigma}_r^{(1)} = \frac{\partial^2\phi^{(1)}}{\partial\bar{z}^2} \qquad \overset{*}{\tau}_1 = \frac{\partial^2\phi^{(1)}}{\partial\xi\partial\bar{z}} \qquad \overset{*}{\sigma}_z^{(1)} = \frac{\partial^2\phi^{(1)}}{\partial\xi^2}.$$

As this form satisfies the now homogeneous equations of equilibrium identically, it follows that $\phi^{(1)}$ is required by the compatibility equation (eqn 47) to satisfy

$$\nabla^4\phi^{(1)} = \frac{\xi}{1-\nu}\frac{\partial^3\bar{T}}{\partial\bar{r}\partial\bar{z}^2}(1,\bar{z}) + 2\frac{\partial}{\partial\xi}(\nabla^2\phi^{(0)}). \tag{50}$$

The boundary conditions on $\phi^{(1)}$ follow from the stress-free boundary conditions, the conditions on $\phi^{(0)}$ and on the particular solution. In particular, these conditions require that

$$\frac{\partial^2\phi^{(1)}}{\partial\bar{z}^2} = 0 \qquad \frac{\partial^2\phi^{(1)}}{\partial\xi\partial\bar{z}} = -\hat{\tau}_b(\bar{z}) \qquad \text{on } \xi = 0, 0 \le \bar{z} \le 1$$

$$\frac{\partial^2\phi^{(1)}}{\partial\xi^2} = 0 \qquad \frac{\partial^2\phi^{(1)}}{\partial\xi\partial\bar{z}} = 0 \qquad \text{on } \bar{z} = 0, 1, 0 \le \xi$$

However, as the shear stress condition on $\xi = 0$ can be integrated to give

$$\frac{\partial\phi^{(1)}}{\partial\xi}(0,\bar{z}) = \frac{\partial\phi^{(1)}}{\partial\xi}(0,0) - \int_0^{\bar{z}} \hat{\tau}_b(x)\,\mathrm{d}x$$

and we have already required that

$$\int_0^1 \hat{\tau}_b(x)\,\mathrm{d}x = 0$$

the boundary conditions can be formulated in the following equivalent form

$$\phi^{(1)} = 0 \qquad \frac{\partial\phi^{(1)}}{\partial\xi} = \bar{z}^2\left[(1-\bar{z})\bar{T}_m(1) - (3-2\bar{z})\int_0^1 Z(1,x)\,\mathrm{d}x\right] + \int_0^{\bar{z}} Z(1,x)\,\mathrm{d}x \qquad \text{on } \xi = 0, 0 \le \bar{z} \le 1 \tag{51}$$

$$\phi^{(1)} = 0 \qquad \frac{\partial\phi^{(1)}}{\partial\bar{z}} = 0 \qquad \text{on } \bar{z} = 0, 1, 0 \le \xi \tag{52}$$

which is analogous to the problem of a laterally loaded plate. In this case, however, the plate is clamped along the sides $\bar{z} = 0, 1$ but hinged on $\xi = 0$ such that the deflection is zero and the slope is prescribed.

This completes the formulation of the second order boundary layer solution. In the following section, we return to the interior solution to look for the form of the corrections that are implied by the existence of the second order boundary layer solution.

SECOND ORDER INTERIOR SOLUTION

Before we can proceed to formulate the matching conditions for the second order boundary layer solution, we must look for the form of the correction to the solution in the interior region. As the correction in the boundary layer was generally of order $\bar{h}$ in comparison with unity, it seems reasonable to expect that the correction to the interior solution to also be of order $\bar{h}$. This may appear to be in contradiction to the observation that the order of the terms neglected when the first order solution was formulated is $0(\bar{h}^2)$. However, the significance of this lies in the fact that the existence of a correction of order $\bar{h}$ depends entirely on the limiting behavior for ξ-large of the boundary layer solution. If the matching conditions do not require such a correction, it does not exist.

Thus, guided by the boundary layer solution, we take the solution in the interior region to have the following form

$$\begin{aligned}
\bar{\sigma}_r(\bar{r}, \bar{z}, \bar{h}) &= \bar{\sigma}_r^{(0)}(\bar{r}, \bar{z}) + \bar{h}\bar{\sigma}_r^{(1)}(\bar{r}, \bar{z}) + O(\bar{h}^2)\\
\bar{\sigma}_\theta(\bar{r}, \bar{z}, \bar{h}) &= \bar{\sigma}_\theta^{(0)}(\bar{r}, \bar{z}) + \bar{h}\bar{\sigma}_\theta^{(1)}(\bar{r}, \bar{z}) + O(\bar{h}^2)\\
\bar{\sigma}_z(\bar{r}, \bar{z}, \bar{h}) &= \bar{\sigma}_z^{(0)}(\bar{r}, \bar{z}) + \bar{h}\bar{\sigma}_z^{(1)}(\bar{r}, \bar{z}) + O(\bar{h}^2)\\
\bar{\tau}(\bar{r}, \bar{z}, \bar{h}) &= \bar{\tau}_0(\bar{r}, \bar{z}) + \bar{h}\bar{\tau}_1(\bar{r}, \bar{z}) + O(\bar{h}^2)\\
\bar{w}(\bar{r}, \bar{z}, \bar{h}) &= \bar{w}_0(\bar{r}, \bar{z}) + \bar{h}\bar{w}_1(\bar{r}, \bar{z}) + O(\bar{h}^2)\\
\bar{u}(\bar{r}, \bar{z}, \bar{h}) &= \bar{u}_0(\bar{r}, \bar{z}) + \bar{h}\bar{u}_1(\bar{r}, \bar{z}) + O(\bar{h}^2)
\end{aligned}$$

When this is substituted into the governing equations (eqns 1–6) and the appropriate limits are taken, there results the following set of equations

$$\frac{\partial \bar{\sigma}_r^{(1)}}{\partial \bar{r}} + \frac{\partial \bar{\tau}_1}{\partial \bar{z}} + (\bar{\sigma}_r^{(1)} - \bar{\sigma}_\theta^{(1)})/\bar{r} = 0 \tag{53}$$

$$\frac{\partial \bar{\sigma}_z^{(1)}}{\partial \bar{z}} + \frac{\partial \bar{\tau}_1}{\partial \bar{r}} + \bar{\tau}_1/\bar{r} = 0 \tag{54}$$

$$\frac{\partial \bar{u}_1}{\partial \bar{r}} = \bar{\sigma}_r^{(1)} - \nu\bar{\sigma}_\theta^{(1)} \qquad \bar{u}^{(1)} = \bar{r}(\bar{\sigma}_\theta^{(1)} - \nu\bar{\sigma}_r^{(1)}) \tag{55, 56}$$

$$\frac{\partial \bar{w}_1}{\partial \bar{z}} = 0 \qquad \frac{\partial \bar{u}_1}{\partial \bar{z}} + \frac{\partial \bar{w}_1}{\partial \bar{r}} = 0 \tag{57, 58}$$

As the transverse normal and shear strain components vanish identically, we obtain

$$\bar{w}_1 = W_1(\bar{r}) \qquad \bar{u}_1 = U_1(\bar{r}) - \bar{z}W_1'(\bar{r})$$

which is the identical in form to the first order solution. Further, the corresponding in-plane stress components are also similar in form to the first order solution except that the temperature terms are absent.

With the $\bar{z}$-dependency of the in-plane stresses now known, we can determine the corresponding transverse shear and normal stresses from the equations of equilibrium (eqns 53, 54). Thus, from eqn (53), the form of the transverse shear stress that satisfies the boundary conditions on $\bar{z} = 0, 1$ is

$$-(1-\nu^2)\bar{\tau}_1 = \bar{z}(1-\bar{z})[(\bar{r}U_1)'/\bar{r}]'$$

as the boundary condition on $\bar{z} = 1$ requires that

$$[(\bar{r}U_1)'/\bar{r} - (\bar{r}W_1')'/2\bar{r}]' = 0.$$

Similarly, we find from eqn (54) that the form of the transverse normal stress that satisfies the boundary condition on $\bar{z} = 0$ is

$$\bar{\sigma}_z^{(1)} = \frac{\bar{z}^2}{2\bar{r}} \cdot \frac{1 - 2\bar{z}/3}{1-\nu^2}\{\bar{r}[(\bar{r}U_1)'/\bar{r}]'\}'.$$

However, as $\bar{\sigma}_z^{(1)}(\bar{r}, \bar{z} = 1) = 0$, then $\bar{\sigma}_z^{(1)} = 0$ and the equation for $U_1(\bar{r})$ becomes

$$[(\bar{r}U_1)'/\bar{r}]' = \text{const.}/\bar{r}.$$

As the left hand side of this equation is essentially the $\bar{r}$-dependency of $\bar{\tau}_1$, we must take the constant to be zero in order to avoid a singularity at $\bar{r} = 0$ with the consequence that $\bar{\tau}_1(\bar{r}, \bar{z}) = 0$.

It then follows that the form of the deflections that is regular at $r = 0$ is

$$U_1 = B_1\bar{r} \qquad W_1' = B_2\bar{r}. \tag{59}$$

Finally, the corresponding in-plane stresses become

$$\bar{\sigma}_r^{(1)} = \bar{\sigma}_\theta^{(1)} = (B_1 - B_2\bar{z})/(1-\nu). \tag{60}$$

Apparently, the correction to the interior solution corresponds to a uniform bending and stretching of the disk without inducing any transverse shear or normal stresses.

With the solution in the interior region now determined to $O(\bar{h}^2)$, we proceed to formulate the matching conditions relating the second order corrections.

MATCHING CONDITIONS FOR THE SECOND ORDER BOUNDARY LAYER SOLUTION

The matching conditions (eqns 29–34) that were studied in the section on first order matching will now be implemented in more detail to determine the requirements on the second order solutions developed above. Thus, on substituting the forms for the boundary layer solution and the solution in the interior region, and noting that the first order in-plane stress components and displacements have the following behavior near the edge $\bar{r} = 1$

$$\bar{\sigma}_r^{(0)}(\bar{r}, \bar{z}) = \bar{\sigma}_r^{(0)}(1, \bar{z}) - \bar{h}\xi\frac{\partial\bar{\sigma}_r^{(0)}}{\partial\bar{r}}(1, \bar{z}) + O(\eta^2) \qquad \text{etc.}$$

we obtain the following requirements

$$L_{\bar{h}\to 0, \bar{r}\text{-fixed}}\left\{[\hat{\sigma}_r^{(0)}(\xi, \bar{z}) - \bar{\sigma}_r^{(0)}(1, \bar{z})]/\bar{h} + \left[\hat{\sigma}_r^{(1)}(\xi, \bar{z}) - \bar{\sigma}_r^{(1)}(\bar{r}, \bar{z}) + \xi\frac{\partial\bar{\sigma}_r^{(0)}}{\partial\bar{r}}(1, \bar{z})\right] + O(\bar{h}, \eta^2/\bar{h})\right\} = 0$$

$$L_{\bar{h}\to 0, \bar{r}\text{-fixed}}\left\{[\hat{\sigma}_\theta^{(0)}(\xi, \bar{z}) - \bar{\sigma}_\theta^{(0)}(1, \bar{z})]/\bar{h} + \left[\hat{\sigma}_\theta^{(1)}(\xi, \bar{z}) - \bar{\sigma}_\theta^{(1)}(\bar{r}, \bar{z}) + \xi\frac{\partial\bar{\sigma}_\theta^{(0)}}{\partial\bar{r}}(1, \bar{z})\right] + O(\bar{h}, \eta^2/\bar{h})\right\} = 0$$

$$L_{\bar{h}\to 0, \bar{r}\text{-fixed}}\{\hat{\sigma}_z^{(0)}(\xi, \bar{z})/\bar{h} + \hat{\sigma}_z^{(1)}(\xi, \bar{z}) + O(\bar{h})\} = 0$$

$$L_{\bar{h}\to 0, \bar{r}\text{-fixed}}\{\hat{\tau}_0(\xi, \bar{z})/\bar{h} + [\hat{\tau}_1(\xi, \bar{z}) - \bar{\tau}_0(\bar{r}, \bar{z})] + O(\bar{h})\} = 0$$

$$L_{\bar{h}\to 0, \bar{r}\text{-fixed}}\left\{[\hat{u}_0(\xi, \bar{z}) - \bar{u}_0(1, \bar{z})]/\bar{h} + \left[\hat{u}_1(\xi, \bar{z}) - \bar{u}_1(\bar{r}, \bar{z}) + \xi\frac{\partial\bar{u}_0}{\partial\bar{r}}(1, \bar{z})\right] + O(\bar{h}, \eta^2/\bar{h})\right\} = 0$$

$$L_{\bar{h}\to 0, \bar{r}\text{-fixed}}\left\{\left[\frac{\partial\hat{w}_1}{\partial\xi}(\xi, \bar{z}) - \frac{\partial\bar{w}_0}{\partial\bar{r}}(1, \bar{z})\right]/\bar{h} + \left[\frac{\partial\hat{w}_2}{\partial\xi}(\xi, \bar{z}) + \frac{\partial\bar{w}_1}{\partial\bar{r}}(\bar{r}, \bar{z}) - \xi\frac{\partial^2\bar{w}_0}{\partial\bar{r}^2}(1, \bar{z})\right] + O(\bar{h}, \eta^2/\bar{h})\right\} = 0$$

Note that, due to our primary interest in the stress distribution, we have written the matching condition for the slope of the transverse deflection rather than the deflection itself.

On inspection of these conditions, it is apparent that we must first verify the limiting behavior of the first order stresses in some detail before we can concentrate on second order matching. Fortunately, we can again make use of Knowles' result quoted earlier to conclude that the first term in each of the conditions for stress matching is transcendentally small as $\bar{h}\to 0$. The corresponding first term in each of the conditions for displacement matching vanishes identically. Thus, if we choose $\eta(\bar{h})$ such that

$$L_{\bar{h}\to 0}\eta^2(\bar{h})/\bar{h} = 0$$

we can concentrate on the second terms in each of the expressions and obtain the required conditions.

In order to establish the behavior of the second order boundary layer stresses and hence evaluate the required limits as $\xi\to\infty$, we define the solution for $\phi^{(1)}$ as the sum of a particular solution $\phi^{(1)}_{\text{part}}$ and a correction, where the particular solution accounts only for the first term on the right hand side of the governing equation. Thus, if we define

$$\phi^{(1)}_{\text{part}} = \frac{\xi}{1-\nu} \cdot \frac{\partial}{\partial \bar{r}}\left\{\int_0^{\bar{z}} Z(\bar{r}, t)\,\mathrm{d}t + \bar{z}^2(1-\bar{z})\bar{T}_m(\bar{r}) - \bar{z}^2(3-2\bar{z})\int_0^1 Z(\bar{r}, t)\,\mathrm{d}t\right\}\Bigg|_{\bar{r}=1}$$

and note that

$$\phi^{(1)}_{\text{part}}(\xi, 0) = \phi^{(1)}_{\text{part}}(\xi, 1) = 0$$
$$\frac{\partial \phi^{(1)}_{\text{part}}}{\partial \bar{z}}(\xi, 0) = \frac{\partial \phi^{(1)}_{\text{part}}}{\partial \bar{z}}(\xi, 1) = 0$$

it is apparent that the problem of determining the correction to $\phi^{(1)}$ is analogous to determining the deflections in a transversely loaded plate. As the "transverse load" is given by

$$2\frac{\partial}{\partial \xi}(\nabla^2 \phi^{(0)}) = 2\left(\frac{\partial \hat{\tau}_0}{\partial \bar{z}} + \frac{\partial \hat{\sigma}_z^{(0)}}{\partial \xi}\right)$$

which has already been shown to be transcendentally small as $\xi \to \infty$, and the effect of the boundary conditions at $\xi = 0$ can be argued to vanish as $\xi \to \infty$, we conclude that

$$\phi^{(1)}(\xi \to \infty, \bar{z}) \to \phi^{(1)}_{\text{part}}(\xi, \bar{z})$$

with the consequence that

$$\hat{\sigma}_r^{(1)}(\xi \to \infty) \to \frac{\partial^2 \phi^{(1)}_{\text{part}}}{\partial \bar{z}^2} + 2\xi\left\{(1-3\bar{z})\left[2\int_0^1 \bar{T}_m(x)x\,\mathrm{d}x - \bar{T}_m(1)\right]\right.$$
$$\left. + 3(1-2\bar{z})\left[\int_0^1 Z(1, x)\,\mathrm{d}x - 2\int_0^1 x\,\mathrm{d}x \int_0^1 Z(x, y)\,\mathrm{d}y\right]\right\} \quad (61)$$

$$\hat{\tau}_1(\xi \to \infty) \to \frac{\partial^2 \phi^{(1)}_{\text{part}}}{\partial \xi \partial \bar{z}} \qquad \hat{\sigma}_z^{(1)}(\xi \to \infty) \to 0. \quad (62)$$

At this stage, all the expressions are available for the evaluation of the terms in the matching conditions. When we make the appropriate substitutions, we find that the terms proportional to ξ vanish identically, and we obtain

$$L_{\bar{h}\to 0, \bar{r}-\text{fixed}}\left[\hat{\sigma}_r^{(1)}(\xi, \bar{z}) + \xi\frac{\partial \bar{\sigma}_r^{(0)}}{\partial \bar{r}}(1, \bar{z})\right] = 0$$

$$L_{\bar{h}\to 0, \bar{r}-\text{fixed}}\left[\hat{\sigma}_\theta^{(1)}(\xi, \bar{z}) + \xi\frac{\partial \bar{\sigma}_\theta^{(0)}}{\partial \bar{r}}(1, \bar{z})\right] = \hat{u}_1(0, \bar{z}) - (1+\nu)F^{(0)}(\infty, \bar{z})$$

$$L_{\bar{h}\to 0, \bar{r}-\text{fixed}}\left[\hat{u}_1(\xi, \bar{z}) - \hat{u}_1(\bar{r}, \bar{z}) + \xi \cdot \frac{\partial \bar{u}_0}{\partial \bar{r}}(1, \bar{z})\right] = \hat{u}_1(0, \bar{z}) - (1+\nu)F^{(0)}(\infty, \bar{z}) - \bar{u}_1(1, \bar{z})$$

$$L_{\bar{h}\to 0, \bar{r}-\text{fixed}}\left[\frac{\partial \hat{w}_2}{\partial \xi}(\xi, \bar{z}) + \frac{\partial \bar{w}_1}{\partial \bar{r}}(\bar{r}, \bar{z}) - \xi \cdot \frac{\partial^2 \bar{w}_0}{\partial \bar{r}^2}(1, \bar{z})\right] = \frac{\partial \bar{w}_1}{\partial \bar{r}}(1, \bar{z}) + \frac{\mathrm{d}}{\mathrm{d}\bar{z}}[\hat{u}_1(0, \bar{z}) - (1+\nu)F^{(0)}(\infty, \bar{z})].$$

On comparing the first result with the corresponding matching requirement, we conclude that

$$\bar{\sigma}_r^{(1)}(1, \bar{z}) = 0.$$

However, from the form of $\bar{\sigma}_r^{(1)}$ determined previously (eqn 60), this can be true only if B_1, $B_2 = 0$. This has the important consequence that the entire second order interior solution must vanish identically. Proceeding with the second result, and noting that $\bar{\sigma}_\theta^{(1)}(1, \bar{z}) = 0$, we obtain

$$\hat{u}_1(0, \bar{z}) = (1+\nu)F^{(0)}(\infty, \bar{z}).$$

This result and the requirement that the second order interior solution vanish identically serves also to satisfy the remaining matching conditions.

At this stage, having established the behavior for ξ-large for both the first order and second order boundary layer solutions, we can proceed to numerically evaluate the stress and displacement distributions. Furthermore, as it has been shown that the second order interior solution vanishes, it can be concluded that the interior solution is accurate to order $\bar{h}^2$ in comparison to unity. However, it should be recalled that the interior solution is essentially that which is obtained from thin plate theory. Thus, we conclude that thin plate theory is accurate to order $\bar{h}^2$, and proceed to determine some numerical results in an attempt to evaluate the relative importance of the stresses in the boundary layer which are not predicted by thin plate theory.

NUMERICAL SOLUTIONS OF THE BOUNDARY LAYER EQUATIONS

In selecting a numerical method for solving the boundary layer problems formulated above, we are guided by the particular simple form of the boundary conditions and the shape of the boundaries themselves to reformulate the problem in finite difference form. Further, as Rushton has recently[10] drawn attention to the advantages of using an iterative technique called "Dynamic Relaxation" for solving plate problems, we adopt the scheme and proceed to describe its use. For a more complete description of the method, the reader should refer to [10] or to Otter[11], who is actually credited with devising the method.

The essential feature of Dynamic Relaxation is the replacement of the equation of equilibrium which depends on space variables only, by the corresponding equation of motion which also contains a viscous damping term. The reason for this apparent increase in complexity is that the equation of motion can be solved explicitly in a step-wise fashion in time more readily than can the equation of equilibrium which requires implicit methods for solution. The magnitude of the viscous damping term is adjusted so that the required static solution emerges from the dynamic solution in a time which corresponds to about 1.5 periods of the lowest mode of free vibration. Essentially, Dynamic Relaxation is analogous to giving the structure an impulsive load and allowing it to vibrate until the kinetic energy is dissipated.

As the governing boundary layer equations have been observed to be similar in form to those governing the transverse deflection of a laterally loaded plate, we proceed to reformulate the problem as a plate problem and so consider the following equation of motion

$$\frac{\partial^2 w}{\partial t^2}+\frac{K}{\Delta t}\cdot\frac{\partial w}{\partial t}=q(\xi,\bar{z})-\nabla^4 w.$$

The factor $K/\Delta t$ is a measure of the damping force and anticipates rewriting the governing equations in finite difference form. The variables w, t are the dimensionless displacement and time respectively. The remaining variables, i.e. the resultant shear force and moments, are identical in form to those of classical plate theory and will not be repeated here.

When the complete set of equations are written in finite difference form, they can be grouped in the following order: (1) deflection-rotation equations, (2) rotation-curvature equations, (3) curvature-moment equations, (4) moment-shear equations and (5) the equation of motion. At any stage of the motion (or iteration), the deflection field is used to generate, in the above-mentioned order, all the additional dependent variables. Once the shear distribution is determined, the equation of motion is used to evaluate the acceleration and hence the velocity for the next time step. Finally, the velocity distribution is used to evaluate the displacement for the next time step. In the first iteration, both the displacement and velocity fields are taken zero. The iteration is continued until the motion effectively ceases.

The principal parameters in the numerical procedure are the spacial intervals $(\Delta\xi, \Delta\bar{z})$, the time step (Δt) and the damping factor. The spacial intervals are generally assumed *a priori*, and taken small enough to accommodate the expected behavior of the deflections. The time step is determined from the condition of numerical stability. As the plate equation is expected to be parabolic in form, we adopt the condition established by Crandall[12] for beams that limits the stable time step to

$$\Delta t \leq (\Delta\bar{z})^2/4$$

and implies that $\Delta\xi$, $\Delta\bar{z}$ are of the same size. It should also be noted that, following Gilles[13], an interlacing mesh is used here in both space and time to improve the accuracy of replacing differentials by differences. For example, rotations are defined in space at points mid-way between both displacements and curvatures, and velocities are defined in time at points mid-way between both displacements and accelerations. Finally, the choice of the damping parameter K is guided by the suggestion of Otter that the motion be slightly underdamped with respect to the first mode of free vibration. Thus, if the critical value of K is identified in terms of the observed period of the first mode (T) by

$$K_{\text{crit}} = 4\pi\Delta t/T$$

we then choose $K \approx 0.70 \times K_{\text{crit}}$.

The first problem to be solved using Dynamic Relaxation is the first order boundary layer problem. In particular, we choose to study the effect of a temperature distribution of constant curvature and take

$$\frac{\partial^2 \bar{T}}{\partial \bar{z}^2}(1, \bar{z}) = \kappa$$

as an example of the simplest temperature distribution that will induce stress. This distribution is very nearly that which would be obtained in a flat plate that was insulated on the back surface and subjected to an impulsive flux of heat that was uniformly distributed over the front surface and constant in time. For the analogous plate problem, we take the transverse load to be unity and hence essentially determine the stress function $\phi^{(0)}$ divided by $-\kappa/(1-\nu)$.

As the displacement distribution is expected to be symmetrical about $\bar{z} = 1/2$, the calculations need only be performed over half the plate (see Fig. 2). In computing the results presented below, the half-plate was divided into 9-segments in the $\bar{z}$-direction and 34-segments in the ξ-direction. The segments were of equal size and the number of segments was chosen so that small changes in the number would produce no appreciable effect on the results. It should be noted that the regularity condition at $\xi \to \infty$ was replaced by the condition that the rotation $(\partial w/\partial \xi)$ vanish for points for which $\xi \approx 34 \times \Delta\xi$. Finally, the damping factor was chosen $K = 0.027$, and the Dynamic Relaxation algorithm was performed 600 times before convergence was judged to be satisfactory.

The results of the calculation are presented as Figs. 3–8. In addition to the stress components,

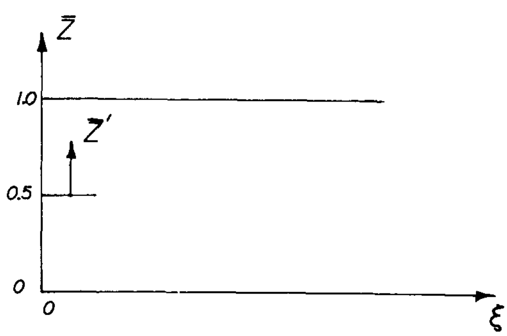

Fig. 2. Coordinate system at edge.

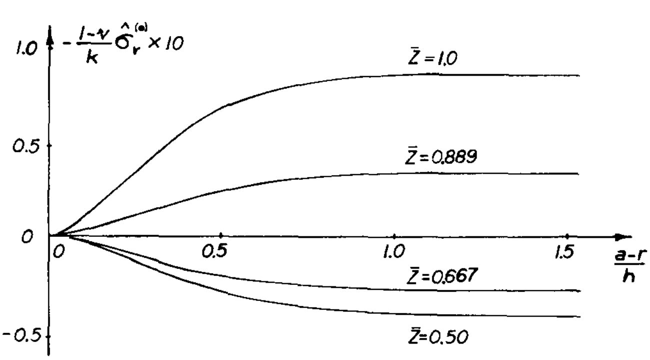

Fig. 3. Radial stress.

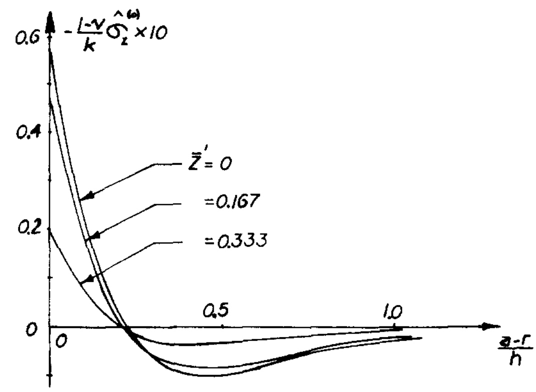

Fig. 4. Transverse stress.

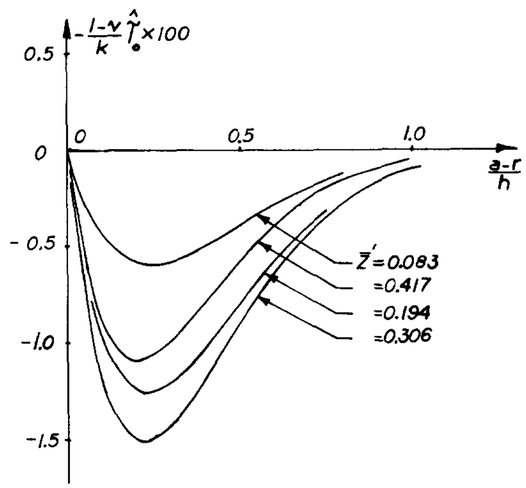

Fig. 5. Shear stress.

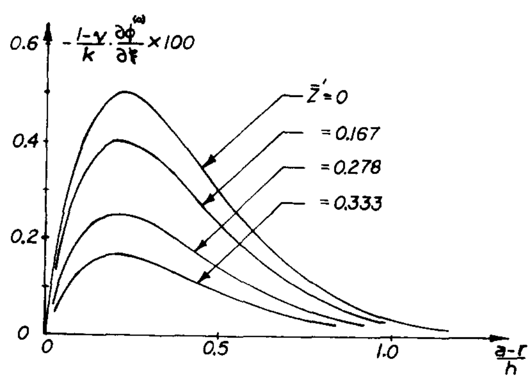

Fig. 6. Radial slope.

the slopes of the stress function are presented so that both the nature of the deflected surface of the analogous plate problem could be visualized and the stress components of the second order correction could be subsequently evaluated. The ξ-component of the analogous plate shear resultant Q_ξ is also presented as it is one of the components of the transverse load of the analogous second order boundary layer problem.

With the stresses $\hat{\sigma}_r^{(0)}$, $\hat{\sigma}_z^{(0)}$ now known, the accompanying "hoop" stress can be evaluated using eqn (37) provided we specify the complete temperature distribution. For purposes of comparison then, let us assume the distribution given by

$$\bar{T} = \bar{T}(\bar{z}) = \kappa\bar{z}^2/2 + C_1\bar{z} + C_0 \tag{63}$$

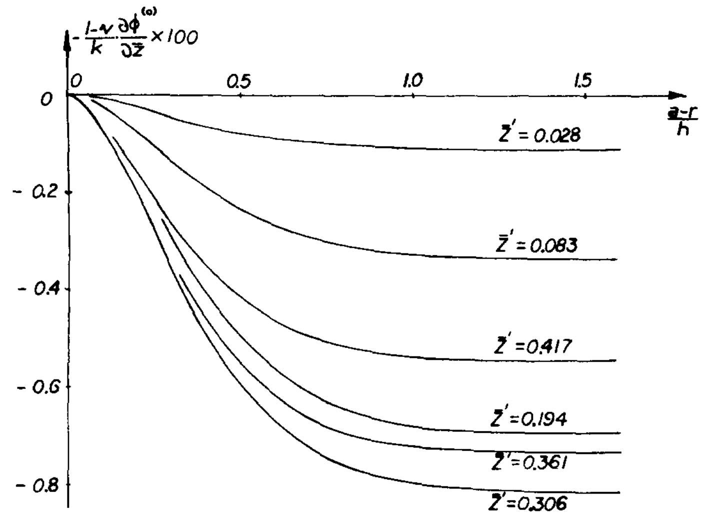

Fig. 7. Transverse slope.

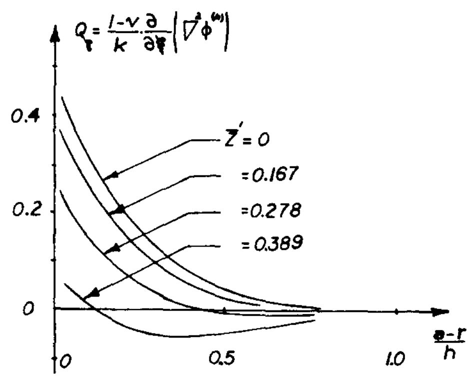

Fig. 8. Transverse loading.

and so obtain

$$\hat{\sigma}_\theta^{(0)} = \nu(\hat{\sigma}_r^{(0)} + \hat{\sigma}_z^{(0)}) - \kappa\,(1 - 6\bar{z} + 6\bar{z}^2)/12.$$

Furthermore, the associated interior stresses become

$$(1-\nu)(\bar{\sigma}_r^{(0)}, \bar{\sigma}_\theta^{(0)}) = -\kappa\,(1 - 6\bar{z} + 6\bar{z}^2)/12. \tag{64}$$

Thus, the "hoop" stress is numerically the largest, varying from its value in the interior to the edge values given by

$$\hat{\sigma}_\theta^{(0)}(0, \bar{z}' = 0) = \kappa\,[1 - 24\nu \times 0.05614/(1-\nu)]/24$$
$$\hat{\sigma}_\theta^{(0)}(0, \bar{z}' = 1/2) = -\kappa\,/12.$$

It is apparent that, at least to first order, the boundary layer is not a region of high stress for a temperature distribution of constant curvature everywhere. However, it should be noted that the behavior of the "hoop" stress in the boundary layer is dependent on the temperature distribution throughout the plate, not just on the curvature at the edge.

With the stress distribution in the boundary layer determined to first order, we now proceed to compute the corrections according to eqns (50)–(52). However, as the main purpose of this exercise is to establish the feasibility of using a modification of the program described above, we shall simplify the problem and limit the temperature distribution to one with zero gradient along the edge $\bar{r} = 1$, and hence take

$$\frac{\partial \bar{T}}{\partial \bar{r}}(1, \bar{z}) = 0.$$

Note that this restriction does not imply that $T = T(\bar{z})$ everywhere, as in the example above. It does imply that the plate is insulated over the edge and is a realistic description of the temperature distribution in some tests for material properties. Thus, the transverse load associated with the analogous plate problem is just the second term on the right-hand side of eqn (50), and is the distribution presented in Fig. 8. For convenience, however, we define

$$\bar{\phi}^{(1)} = (1-\nu)\phi^{(1)}/\kappa$$

and hence reduce eqns (50), (51) to

$$\nabla^4\bar{\phi}^{(1)} = 2Q_\xi \qquad \frac{\partial\bar{\phi}^{(1)}}{\partial\xi}(0, \bar{z}) = (1-\nu)\bar{z}^2(1-\bar{z})^2/24.$$

The analogous plate problem is then to determine the displacement due to a distributed load that is an even function of $\bar{z}'$ and which decays rapidly with distance from the edge $\xi = 0$. The plate is clamped along $\bar{z} = 0, 1$, and is hinged at $\xi = 0$ so that the deflection is zero and the slope is also an even function of $\bar{z}'$. As before, the regularity condition at $\xi \to \infty$ is replaced by the condition that the rotation vanish along $\xi \approx 34 \times \Delta\xi$.

The results of calculations made using the program described above with the same program parameters and taking $\nu = 0.3$ are presented as Figs. 9–11. It should be noted that there is no difficulty in modifying the program to account for the hinged boundary condition. It is apparent from the figures that the corrections are relatively small and decay rapidly to zero as ξ-increases. For a complete evaluation of the second order stress distribution, one must substitute these components into eqn (44) along with the slopes given above (Figs. 6, 7) and the corresponding expressions for $\hat{\sigma}_r^{(b)}$, $\hat{\tau}_b$. For purposes of comparison, we again evaluate the particular solution for the special case of a temperature distribution varying with $\bar{z}$-only (eqn 63), and so obtain

$$\hat{\sigma}_r^{(b)} = 0 \qquad \hat{\tau}_b = -\kappa\bar{z}(1-\bar{z})(1-2\bar{z})/12.$$

The composite solution is not shown, but can readily be constructed from these results. Though it can be seen that these corrections to the first order boundary layer stresses are not negligible, they do not alter the conclusion that the boundary layer is not a region of stress concentration for a temperature distribution that is independent of $\bar{r}$. Thus, the maximum tensile and compressive stresses in a plate that is heated uniformly over its surface are accurately predicted by classical plate theory, and the practice of using these predictions to correlate experimental thermal shock data is therefore justified.

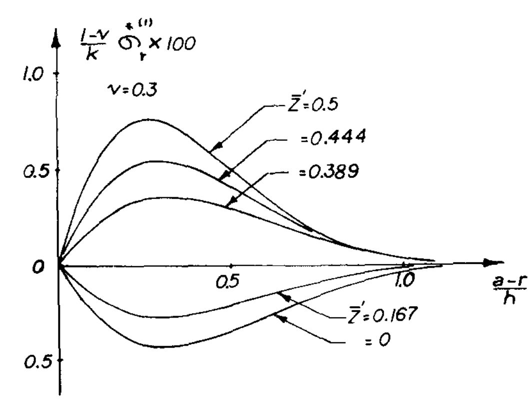

Fig. 9. Radial stress component.

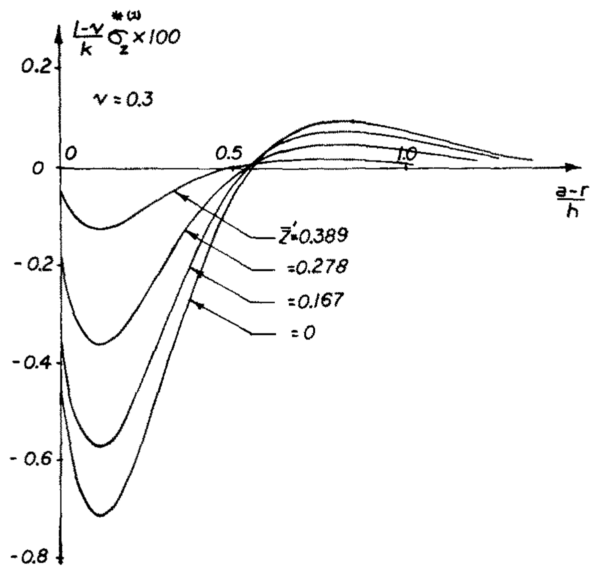

Fig. 10. Transverse stress component.

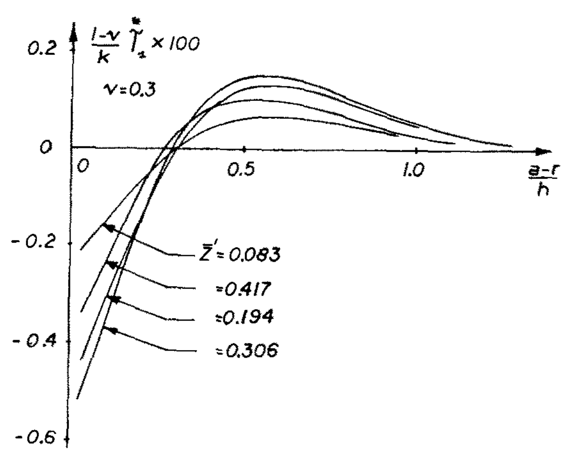

Fig. 11. Shear stress component.

SUMMARY AND DISCUSSION

From the point of view of the analyst, the most important result obtained above is that solutions given by classical plate theory are accurate to order $(h/a)^2$ in comparison to unity in the interior of the plate. Further, it is justifiable to require the force and moment stress resultants to both vanish at $r = a$ in order to obtain plate solutions for the stress-free boundary condition. As far as boundary layer effects are concerned, the classical plate boundary conditions on displacement are sufficient to establish the solution for the clamped plate, as there is no boundary layer in the first order solution. However, for the stress-free boundary condition, a boundary layer does exist and the stress components in the layer are all of the same order and must be obtained numerically.

Before attempting a numerical solution of the boundary layer equations, some important properties of the solution can be obtained from an examination of the governing equations. As far as the first order problem is concerned, the stress distribution depends on the curvature of the temperature distribution evaluated at the edge. However, the "hoop" stress component depends on the temperature distribution throughout the plate. For the constant curvature case studied numerically above, the "hoop" stress is numerically the largest. In general, however, this is likely to be the case as the radial stress must vanish at the free edge and the expression for the "hoop" stress is dominated by the contribution from the temperature integrals over the whole plate. This is probably the cause of the radial crack mode of failure which is often observed.

The second order boundary layer stress distribution depends on the curvature of the temperature distribution at the edge, the temperature distribution throughout the plate and also on the edge distribution of the radial temperature gradient. In the numerical example above, the temperature distribution was independent of r and the stress distribution decayed with distance (ξ) from the edge. It should be noted, however, that this is generally not the case, and, following eqns (61), (62), one should expect the limiting behavior of the radial stress and the shear stress to be linear in ξ and constant respectively.

The numerical solution of the boundary layer equations was obtained using the Dynamic Relaxation algorithm. The program that was written for the first order equivalent plate problem was easily modified to account for the hinged boundary condition of the second order equivalent plate problem. It should be noted that this program could also be modified to account for a completely general, axisymmetric temperature distribution.

The solution process formulated above is capable of being extended to any order of approximation, but it is questionable whether such effort would be justified. Should more accuracy be required than is obtained in the above analysis, it is probable that it would be more expedient to do the whole problem numerically using an existing Finite Element program. However, it is not inexpensive to investigate boundary layer phenomena using the Finite Element Method, so that the solution procedure described above does have a distinct economic advantage in the analysis of thin disks.

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